Arbitrage Pricing *If financial markets are efficient, then assets with the same known income streams will have the same market values.*

The Miracle of Compound Interest: $FV = PV\ (1+R)^N$.

Compounding M times per year: $FV = PV\ (1+R/M)^{M \cdot N}$.

Instantaneous Compounding: $FV = PV\ e^{R \cdot N}$.

12% Interest for 10 Years					
Year	Frequency of Compounding				
	Annual	Quarterly	Monthly	Daily	Instantaneous
0	100.00	100.00	100.00	100.00	100.00
1	112.00	112.55	112.68	112.75	112.75
2	125.44	126.68	126.97	127.12	127.12
3	140.49	142.58	143.08	143.32	143.33
4	157.35	160.47	161.22	161.59	161.61
5	176.23	180.61	181.67	182.19	182.21
6	197.38	203.28	204.71	205.42	205.44
7	221.07	228.79	230.67	231.60	231.64
8	247.60	257.51	259.93	261.13	261.17
9	277.31	289.83	292.89	294.41	294.47
10	310.58	326.20	330.04	331.95	332.01

Pure Discount Bond — FV after N periods

$$PV = FV / (1+R)^N$$

Consol — X per period forever

$$PV = X/(1+R) + X/(1+R)^2 + \cdots = X / R$$

Mortgage — X per period for N periods

$$PV = X/(1+R) + X/(1+R)^2 + \cdots + X/(1+R)^N$$

Coupon Bond — X per period for N periods plus F after N periods

$$PV = X/(1+R) + X/(1+R)^2 + \cdots + X/(1+R)^N + F/(1+R)^N$$

Interest Rate Risk

Duration: *the percentage change in market value of an asset caused by a one percentage point change in interest rates.*

For a pure discount bond, the duration is roughly equal to the time to maturity.

Proof: $PV = FV / (1+R)^N = FV (1+R)^{-N}$.

$MPV/MR = -N FV / (1+R)^{N+1} = -N FV (1+R)^{-N-1}$.

$MPV/MR / PV = -N / (1+R)$.

The original definition of duration was the weighted average of the time to loan repayment. The essential point to be appreciated is that a longer time until a loan is repaid (and a greater duration) leads to a greater sensitivity to change in interest rates. We refer to this sensitivity as *interest rate risk*. Interest rate risk is a major concern even for government bonds with no chance of default.

The following table demonstrates the relationship between interest rate risk and bond duration. While the calculations are a little difficult, the duration of a 30 bond is about 9, the duration of a 5 year bond is about 4, and the duration of a 1 year bond is about 1. The duration is less than the time to maturity because the coupon stream represents a substantial fraction of the present value for a long-term bond and the coupons are received before maturity.

Bond Value vs. Discount Rate and Bond Maturity

	30 Year 10% Coupon Bond			**5 Year 10% Coupon Bond**			**1 Year 10% Coupon Bond**		
R	Coupons	Principal	Bond	Coupons	Principal	Bond	Coupons	Principal	Bond
0.05	1537.25	231.38	1768.62	432.95	783.53	1216.47	95.24	952.38	1047.62
0.06	1376.48	174.11	1550.59	421.24	747.26	1168.49	94.34	943.40	1037.74
0.07	1240.90	131.37	1372.27	410.02	712.99	1123.01	93.46	934.58	1028.04
0.08	1125.78	99.38	1225.16	399.27	680.58	1079.85	92.59	925.93	1018.52
0.09	1027.37	75.37	1102.74	388.97	649.93	1038.90	91.74	917.43	1009.17
0.10	942.69	57.31	1000.00	379.08	620.92	1000.00	90.91	909.09	1000.00
0.11	869.38	43.68	913.06	369.59	593.45	963.04	90.09	900.90	990.99
0.12	805.52	33.38	838.90	360.48	567.43	927.90	89.29	892.86	982.14
0.13	749.57	25.57	775.13	351.72	542.76	894.48	88.50	884.96	973.45
0.14	700.27	19.63	719.89	343.31	519.37	862.68	87.72	877.19	964.91
0.15	656.60	15.10	671.70	335.22	497.18	832.39	86.96	869.57	956.52
0.16	617.72	11.65	629.37	327.43	476.11	803.54	86.21	862.07	948.28
0.17	582.94	9.00	591.94	319.93	456.11	776.05	85.47	854.70	940.17
0.18	551.68	6.97	558.66	312.72	437.11	749.83	84.75	847.46	932.20
0.19	523.47	5.41	528.88	305.76	419.05	724.81	84.03	840.34	924.37
0.20	497.89	4.21	502.11	299.06	401.88	700.94	83.33	833.33	916.67

Question: Are long-term government bonds a safe investment?

Intertemporal Substitution

Indifference curve: *A set of points with the same utility.*

Budget constraint: *The set of possible choices given an endowment.*

From microeconomics, we are familiar with the notion of marginal utility. The standard two-good utility maximization diagram shows how an agent would choose the consumption amounts for two goods that maximize that agent's utility subject to his or her budget constraint. At that point (unless it is a corner solution), the marginal utilities are in the same ratio as the prices and the ratio of the prices is the slope of the budget constraint. The standard microeconomic analysis goes on to define and illustrate the income and substitution effects. The latter exercise gives us insight into how changes in prices and income affect consumption patterns.

In our intertemporal substitution diagram, the interest rate functions as the relative price of money now compared with money in the future. A subsequent discussion will consider the possibility that prices also differ between the present and the future, but for now we will assume that they are the same. We will review the analysis from microeconomics with the intention of gaining insight into how interest rates and income affect agents' saving and borrowing decisions under the assumption that their endowment is simply a cash payment in either the present or the future. We will also discuss agents who have to earn money by working.

Cases with straight-line budget constraints

Endowment in the present
Endowment in the future
Endowment in the present and in the future

Pure income effect
Pure substitution effect
Typical income/substitution effect diagram

Cases with "kinks"

No access to capital markets or capital markets do not exist
Saving possible, but not borrowing
Corner solutions

Curved production possibility frontiers

Time invested in education
Gains from trade

Question: Can you match these topics with the diagrams on the next two pages?

Real Interest Rates

Fundamental Equation $R = \pi + r$

R	nominal interest rate
π	inflation rate
r	real interest rate

Mathematical theorem:

If a and b are near zero, then $(1+a)(1+b) \approx 1+a+b$.
If c and b are near zero, then $(1+c)/(1+b) \approx 1+c-b$.

Proof #1: $(1+a)(1+b) \approx 1+a+b+ab$, and ab is very small. Proof #2: $(1.06)(1.10) = 1.1660 \approx 1.16$.
Proof #3: $(1.01)(1.02) = 1.0302 \approx 1.03$. Proof #4: $(1.10)/(1.04) = 1.0577 \approx 1.06$.

Economics Application:

If the nominal interest rate is R and the inflation rate is π, then a bank deposit gains real purchasing power at the real interest rate of $r \approx R - \pi$. Reasonable savers care about the real interest rate, not the nominal interest rate.

Investments in real physical assets are an alternative investment for agents who might otherwise put their money in a bank account. Real physical investment is also how people who borrow money from banks expect to repay the bank. The real return to physical investment depends on technology, which does not change when inflation changes. We are thus led to the possibility that (1) the real rate of interest is anchored to the real rate of return on phyical assets and (2) the real rate of interest is likely to be relatively stable. Economists typically think of a real interest rate of 0%, 1%, or 2%. The precise value and the extent of its stability are difficult measurement problems and the subject of considerable debate.

Budget Constraint for Intertemporal Substitution with Inflation

W_1	present value of total wealth	$w_1 = W_1 / p_1$	present value of real wealth
p_1	price level in period 1	p_2	price level in period 2
y_1	real endowment in period 1	y_2	real endowment in period 2
x_1	real consumption in period 1	x_2	real consumption in period 2

The present value of your endowments must equal (or exceed) the present value of your consumption, so we have

$$W_1 = p_1 y_1 + p_2 y_2 / (1+R) \qquad W_1 = p_1 x_1 + p_2 x_2 / (1+R)$$

By the definition of the inflation rate π, we have $p_2 = p_1 (1+\pi)$ so

$$w_1 = y_1 + y_2 (1+\pi) / (1+R) \qquad w_1 = x_1 + x_2 (1+\pi) / (1+R)$$

Invoking the real interest rate definition $r = R - \pi$ and the mathematical theorem,

$$w_1 = y_1 + y_2 / (1+r) \qquad w_1 = x_1 + x_2 / (1+r)$$

The bottom line here is that the budget constraint recognizing inflation uses the real interest rate in place of the nominal interest rate. To correctly decide between using the nominal or real interest rates, you need to think clearly about whether you are comparing real or nominal quantities in the two periods.

Why do Firms Borrow Money to Make Physical Investments?

The intertemporal substitution story with a straight-line budget constraint explains why people save money to finance future consumption. The technology for this savings process is linear; one dollar saved in the present yields 1+R dollars in the future regardless of the amount of savings or the endowment point. We can take into account a production technology by adding a production possibility frontier to this story. It will then be the case that the trade-off between present and future production is nonlinear; the additional production in the future secured by an additional dollar of current investment depends on the initial mix of current and future production. In equilibrium, firms will invest so that marginal returns and marginal utilities are equated in the usual way.

The Project A,B,C story is a simpler explanation of how firms view marginal returns. It simplies the problem by working with a fixed rate of interest on borrowed money. The utility function then does not enter the analysis because the invested funds come from a large pool of loanable funds and one firm's actions do not affect the pool's marginal utilties. In this situation, the firm can turn its attention away from the utilities of investors and simply maximize the dollar value of the firm. That is, dollars are a good measure of the utility relevant to the financial markets as a whole.

The firm might adopt one of two decision rules:

Invest in only those projects where the present value of the current and future benefits at least equals the present value of the current and future costs.

Invest in only those projects where the rate of return on the project at least equals the interest rate on the loan to finance the project.

We will find by numerical example that these two rules are two sides of the same coin. They lead to the same decisions in our framework. For example, suppose that Project A returns \$X in N years and costs \$C to finance with an R% loan. The present value of the benefits is $X/(1+R)^N$ and the present value of the costs is C. The rate of return is that number Q such that $C = X/(1+Q)^N$. We will try to gain the insight that Q is greater than or equal to R in precisely those circumstances where $X/(1+R)^N$ is greater than or equal to C.

The most challenging issue raised by this discussion is this: If a firm finds it profitable to invest in Project A today because it has a 12% rate of return and the interest rate on borrowed funds is 10%, then why didn't the firm make the investment yesterday? That is, how can there be unexploited profit opportunities. Our usual analysis is that, given free markets, there is no point in looking for such situations because the markets will immediately remove them.

Three additional ideas may be involved here. First, there are almost certainly risks involved in investments and considerations of risk will complicate the analysis. There may well be unexploited risky investments available. Second, in a dynamic economy with changing technology, the array of available investments changes so that today's investment opportunities may not have existed previously or they may not have been profitable under previous technology. For example, investment in VCR tape rental outlets was not technologically feasible 20 years ago.

A third idea arises from another challenge. If Project A currently has a higher rate of return than Projects B and C (even though all three meet the rule above), why would a firm ever invest in Projects B and C instead of replicating Project A? To address this issue we can introduce economies of scale. There may be technical limits to how many Project A's a firm can create (think of McDonald's), but even more importantly there may be limits to how many Project A's the human management can effectively supervise (think of all the bankrupt small restaurant chains).

What we have discovered from this discussion is that the decision to invest in production is far more complicated than the decision to invest in financial assets because production technology has many interesting, but challenging features. We will, therefore, leave some of those issues for other courses.